Pseudospectral methods and transient growth in hydrodynamic stability of parallel flows

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For systems described by non-normal matrices or linear operators, analysis of linear eigenvalue stability about fixed points alone is insufficient, as such systems can exhibit extremely large *non-modal* growth that undermines the original linear assumption. So-called pseudospectral analysis is helpful for analyzing such systems. The particular case investigated here is the hydrodynamic stability in parallel flows. Pseudospectra, eigenvalue perturbation plots, and transient growth results are presented for Poiseuille and Couette flows at various Reynolds numbers, and for various streamwise and spanwise wave numbers.

I. Introduction

In even the most elementary problems in boundary layer studies, there is a large difference between the best experimental observations of transition to turbulence and the behavior expected from linear stability analysis using eigenvalues. That is, boundary layer transition is often observed for much lower Reynolds numbers than is expected according to the eigenvalue analysis, or - in the case of Couette flow - even when transition is not predicted at all regardless of Reynolds number. This is often referred to as "subcritical transition to turbulence."

In many cases, this phenomenon can be accounted for by considering the transient growth which is not apparent when only considering eigenvalue stability in non-normal systems[a]. In such systems, even when eigenvalue analysis indicates that all eigenmodes asymptotically decay, the solutions themselves may grow by several orders of magnitude before time is sufficiently large that they decay to zero – depending, of course, on initial conditions [2]. Such non-normal systems are numerous, and the technique discussed in this project for understanding transient growth in fluid stability also applies directly to problems in fields such as control theory, lasers, ecology, atmospheric science, Markov chains, and more.

This project serves to explain pseudospectra and how they can estimate transient growth in these systems.

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[a]That is, systems where the eigenfunctions are not orthogonal to one another (though they are indeed linearly independent). This occurs for matrices for which $A^*A \neq AA^*$

II. Definitions of Pseudospectra

Before we utilize the pseudospectra to aid in our analysis of the transient growth in non-normal systems, we first formally define them.

We define the pseudospectra using the following defintions proposed by Trefethen[1], where z is an ϵ-pseudoeigenvalue. The pseudospectra is the set of z in the complex plane.

$$||(z\mathbf{I} - \mathbf{A})^{-1}|| \geq \epsilon^{-1} \tag{1}$$

It is clear in (1) that in the limit $\epsilon = 0$, z indeed become the actual eigenvalues of $\mathbf{A}$, as the resolvent norm tends to infinity.

This primary definition is most useful in that it facilitates easy creation of pseudospectral contour plots. To do this, we simply create a mesh grid of z over a region of interest in the complex plane among the eigenvalues of $\mathbf{A}$. At each point in the grid, we calculate the norm of the resolvent, and arrive with a value of ϵ, and a contour plot can be created easily.

An alternative (though equivalent) definition offers a unique perspective by more directly describing eigenvalue sensitivity:

$$det\,(z\mathbf{I} - (\mathbf{A} + \mathbf{E})) = 0, \tag{2}$$

where $\mathbf{E}$ is a random matrix with mean value 0 and norm $||\mathbf{E}|| \leq \epsilon$. Here, we simply say that z is an ϵ-pseudoeigenvalue of $\mathbf{A}$ if it is itself the eigenvalue of the perturbed matrix $\mathbf{A}+\mathbf{E}$. Though this project concentrates on the transient growth in fluid flows, this definition of the eigenvalue sensitivity phenomenon explains an alternate mechanism for transition. Say, perhaps, that some experimental imperfection makes the operator itself not quite right - and that the system is actually represented by the perturbed operator $\mathbf{A}+\mathbf{E}$. Here, the physical system may indeed have exponential (eigenvalue-based) growth, though the "theoretical" operator $\mathbf{A}$ is eigenvalue stable.

III. Connection Between Pseudospectra and Transient Growth

Trefethen [1] provides the following simple lower bound for the transient growth in terms of the abscissa of the ϵ-pseudoeigenvalues z:

$$\max||e^{t\mathbf{A}}|| \geq \frac{\max(\text{real}(z))}{\epsilon} \tag{3}$$

Here, we see that a lower bound for the maximum energy growth for all time can be described entirely in terms of the pseudospectra. All we must do is identify the ϵ-pseudoeigenvalue which extends furthest into the unstable half-plane, and divide that distance by the value of ϵ that generated this z. As useful as this quick estimate is, will see in later analysis that this lower bound is actually rather conservative, and much larger growth factors can be experienced by computing the actual evolution of the matrix exponential in time.

IV. Hydrodynamic Stability of Parallel Flows

For this project, we consider the stability of plane Poiseuille flow, which is a two-dimensional flow between two stationary plates two unit distances apart. The flow is pictured below in Figure 1.

This flow is driven by a pressure gradient in the x direction, and gains its parabolic shape from the effects of viscosity and the "no-slip" boundary conditions at each wall.

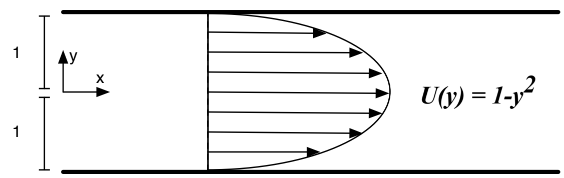


Figure 1. Poiseuille Flow Diagram

We will also briefly consider Couette flow, which can be described as the two-dimensional flow between a stationary plate and a plate two unit distances away which is moving with unit velocity. Couette flow - we will soon show - is eigenvalue stable for all combinations of Reynolds numbers and streamwise and spanwise wave numbers, yet still exhibits striking transient growth.

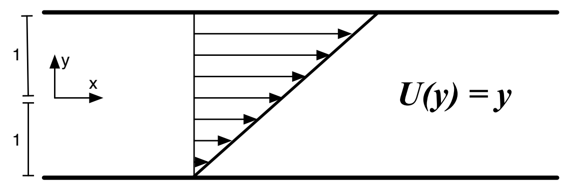


Figure 2. Couette Flow Diagram

A. Orr-Sommerfeld Derivation

Whereas in all forms of eigenvalue analysis we linearize about a "fixed point", here we linearize about the laminar flow basic state solution to the problem.

To derive the Orr-Sommerfeld operator we use to describe the evolution of disturbances in parallel flows, we begin with the incompressible Navier Stokes equations. Equation 4 describes momentum balance, and (5) is the mass balance equation.:

$$\frac{\partial u_i}{\partial t} = -u_j \frac{\partial u_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{1}{\mathrm{Re}} \nabla^2 u_i \tag{4}$$

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{5}$$

Next, we introduce small perturbations, denoted by lower-case primed quantities u_i', to the basic state variables U_i, where the basic state is itself a solution to the Navier-Stokes equations.

$$u_i = U_i + u_i' \tag{6}$$

$$p_i = P_i + p_i' \tag{7}$$

Now, we substitute these perturbed quantities into (4) and (5), and subtract the terms containing only basic state variables, since these comprise a known solution. This yields:

$$\frac{\partial u_i'}{\partial t} = -U_j \frac{\partial u_i'}{\partial x_j} - u_j' \frac{\partial U_i}{\partial x_j} - \frac{\partial p'}{\partial x_i} + \frac{1}{\mathrm{Re}} \nabla^2 u_i' - u_j' \frac{\partial u_j'}{\partial x_j} \tag{8}$$

$$\frac{\partial u_i'}{\partial x_i} = 0 \tag{9}$$

For linear analysis, we consider small disturbances, neglecting higher order terms in the disturbance quantities.. Also, for this project, we will consider parallel flows, where $U_i = U(y)\hat{i}$ only. Now, the disturbance equations are reduced to

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + vU' = -\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}} \nabla^2 u \tag{10}$$

$$\frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y} + \frac{1}{\mathrm{Re}} \nabla^2 v \tag{11}$$

$$\frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} = -\frac{\partial p}{\partial z} + \frac{1}{\mathrm{Re}} \nabla^2 w \tag{12}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{13}$$

where we have eliminated the prime notation for convenience.

Taking the divergence of (10) through (12) and using continuity yields a way to eliminate p:

$$\nabla^2 p = -2U' \frac{\partial v}{\partial x} \tag{14}$$

After futher manipulation of (10) through (12) with (14), we reduce the set of equations to a single operator on the normal velocity v.:

$$\left[\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \nabla^2 - U'' \frac{\partial}{\partial x} - \frac{1}{\mathrm{Re}} \nabla^4 \right] v = 0 \tag{15}$$

Now, we consider wavelike disturbances of the form:

$$v(x, y, z, t) = \hat{v}(y) e^{i(\alpha x + \beta z - \omega t)} \tag{16}$$

Substituting (16) into (15), we obtain the classical Orr-Sommerfeld Equation:

$$\left[(-i\omega + i\alpha U) \left(\mathcal{D}^2 - k^2 \right) - i\alpha U'' - \frac{1}{\mathrm{Re}} \left(\mathcal{D}^2 - k^2 \right)^2 \right] \hat{v} = 0 \tag{17}$$

$$\hat{v}(\pm 1) = D\hat{v}(\pm 1) = 0$$

This is an eigenvalue problem for ω. It is conventional to instead create an eigenvalue problem for the phase speed c, consider $\omega = \alpha c$:

$$\left[(U-c)(\mathcal{D}^2-k^2) - U'' - \frac{1}{i\alpha\mathrm{Re}}(\mathcal{D}^2-k^2)^2\right]\hat{v} = 0 \tag{18}$$

It can now be shown through Squire's transformation, which compares the above 3-D case for α and β with the 2-D case where $\beta = 0$, that each three dimensional Orr-Sommerfeld mode exactly corresponds to a two dimensional mode at a *lower* Reynolds number. Thus, it is said that we only need to consider α to determine critical behavior:

$$\left[(U-c)(\mathcal{D}^2-\alpha^2) - U'' - \frac{1}{i\alpha\mathrm{Re}}(\mathcal{D}^2-\alpha^2)^2\right]\hat{v} = 0 \tag{19}$$

It will be shown later that this is not entirely the best approach, due to the large transient growth of "three-dimensional" perturbations, with $\beta \neq 0$. Thus, going forward, we will use the form of the Orr-Sommerfeld equation shown in (18).

B. Discretization of the Orr-Sommerfeld Operator

Symbolically, this is the limit of our analysis with regards to the Orr-Sommerfeld operator. To obtain meaningful analysis of the stability of flows, we must discretize the operator in preparation for numerical analysis in MATLAB.

Pure finite-difference methods can be used successfully, but greater accuracy can be obtained with smaller matrices using a spectral collocation method based on Chebyshev polynomials [7].

The Chebyshev polynomials are a set of orthogonal basis functions, and are given as follows:

$$T_n(y) = \cos(n \arccos(y))$$

First, we expand our velocity and derivatives in series of these basis functions:

$$\hat{v}(y) = \sum_{n=0}^{N} a_n T_n(y) \tag{20}$$

$$D^2\hat{v}(y) = \sum_{n=0}^{N} a_n T_n''(y) \tag{21}$$

$$D^4\hat{v}(y) = \sum_{n=0}^{N} a_n T_n''''(y) \tag{22}$$

and substitute into the Orr-Sommerfeld operator (18):

$$\begin{aligned}\left(U(y)k^2 - U''(y) - \frac{k^4}{i\alpha Re}\right)\sum_{n=0}^{N} a_n T_n(y) \quad &+ \quad \left(U(y) + \frac{2k^2}{i\alpha Re}\right)\sum_{n=0}^{N} a_n T_n''(y) - \frac{1}{i\alpha Re}\sum_{n=0}^{N} a_n T_n''''(y) \\ &= \quad c\left(\sum_{n=0}^{N} a_n T_n''(y) - k^2 \sum_{n=0}^{N} a_n T_n(y)\right)\end{aligned} \tag{23}$$

The clever trick is introduced by discretizing the y domain at the extrema of the N-th Chebyshev polynomial:

$$y_j = \cos\left(\frac{j\pi}{N}\right)$$

Using this domain, the derivatives of the polynomials can be expressed recursively as follows:

$$T_n^{(k)}(y_j) = 2nT_{n-1}^{(k-1)}(y_j) + \frac{n}{n-1}T_{n-1}^{(k)}(y_j)$$

This forms an eigenvalue problem for the phase speed c, with eigenvectors as the expansion coefficients $\vec{a} = a_n$:

$$\mathbf{A}\vec{a} = c\mathbf{B}\vec{a}$$

where $\mathbf{A}$ and $\mathbf{B}$ are easy-to-calculate functions of α, β, Re, and the values of the Chebyshev functions T_0 through T_4 evaluated at y_j. The first two and last two rows of $\mathbf{A}$ and $\mathbf{B}$ are used to impose the boundary conditions.

The fact that we're now using a_n as the modes makes plotting the Orr-Sommerfeld velocity modes themselves less straight-forward than if we were using y_n as in a finite difference method, but the benefits in discretization are worthwhile.

V. Results

A. Classical Eigenvalue Stability

Using the above discretization of the Orr-Sommerfeld operator, and the known basic state velocity profiles for Poiseuille and Couette flows, we can then conduct a classical eigenvalue stability analysis, iteratively solving for the eigenvalues for a range of the parameters Re and α.

Shown in Figure 3 is the classical 2-D eigenvalue stability boundary for Poiseuille flow. The contours shown are contours of c_i, the imaginary part of the complex phase speed c. Inside the unstable region, c_i is negative. Due to the form of the disturbance defined by (16), this negative imaginary part coincides with exponential growth. Though this is contrary to the usual definition of stability, where we seek unstable solutions as those with the *real part* of the eigenvalue *greater* than zero), this is the conventional notation in the fluids literature, and we will continue as such throughout the results below.

Here, we see that for Reynolds numbers less than 5772, we should not expect eigenvalue instability. Though it remains true that all solutions at these Reynolds numbers will decay to zero asymptotically, subsequent sections will show that transient growth is significant.

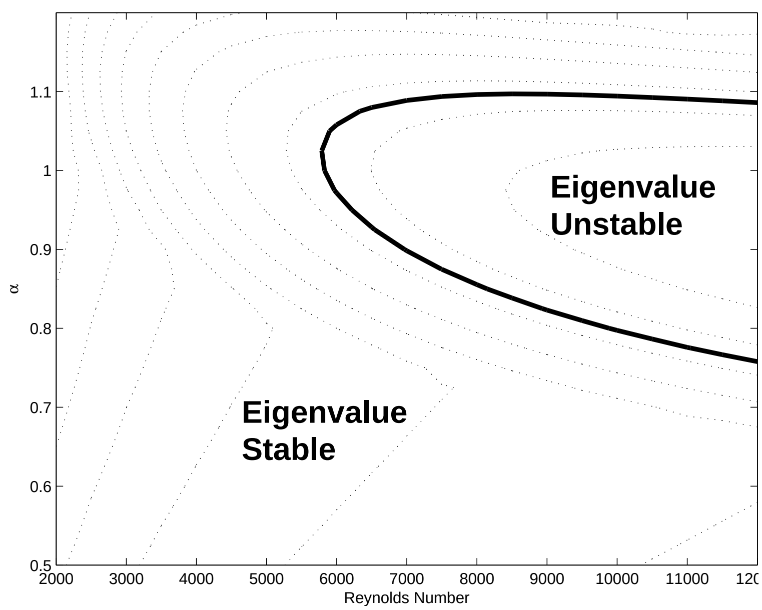


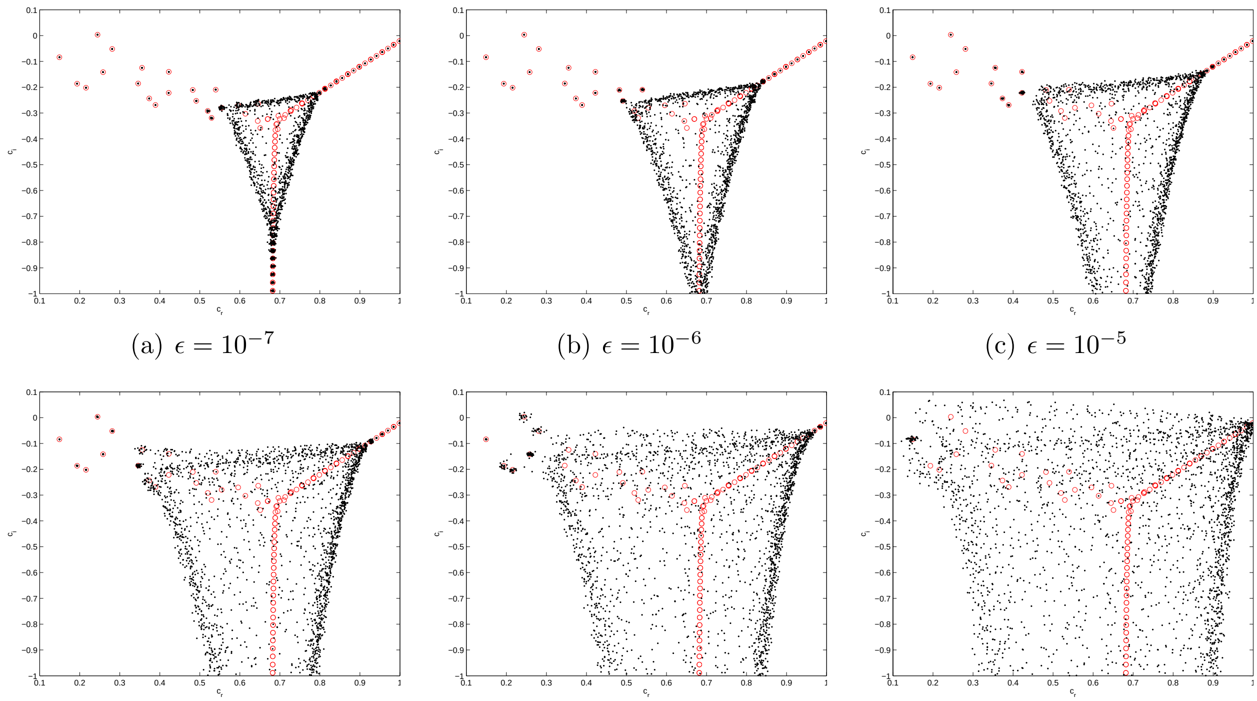
Figure 3. Eigenvalue Stability of Poiseuille Flow

A similar procedure can be conducted for Couette flow, but there is no stability boundary, as this flow is eigenvalue stable for all parameter combinations[7].

B. Eigenvalue Perturbations

To begin illustrating the concept of pseudospectra and eigenvalue sensitivity, we have taken the Orr-Sommerfeld operator for Poiseuille and Couette flows, and perturbed them with various random matrices of norm ϵ between 10^{-7} and 10^{-2}. The eigenvalues of these perturbed matrices, in accordance with the definition in (2), are ϵ-pseudoeigenvalues, and illustrate the extreme sensitivity of the operator to perturbations. The results are shown below in

Figures 4 and 5. The red dots are the unperturbed eigenvalues of the operator, and the black dots are twenty sets of eigenvalues of randomly perturbed operators, superposed on one another.



(a) $\epsilon = 10^{-7}$ (b) $\epsilon = 10^{-6}$ (c) $\epsilon = 10^{-5}$

(d) $\epsilon = 10^{-4}$ (e) $\epsilon = 10^{-3}$ (f) $\epsilon = 10^{-2}$

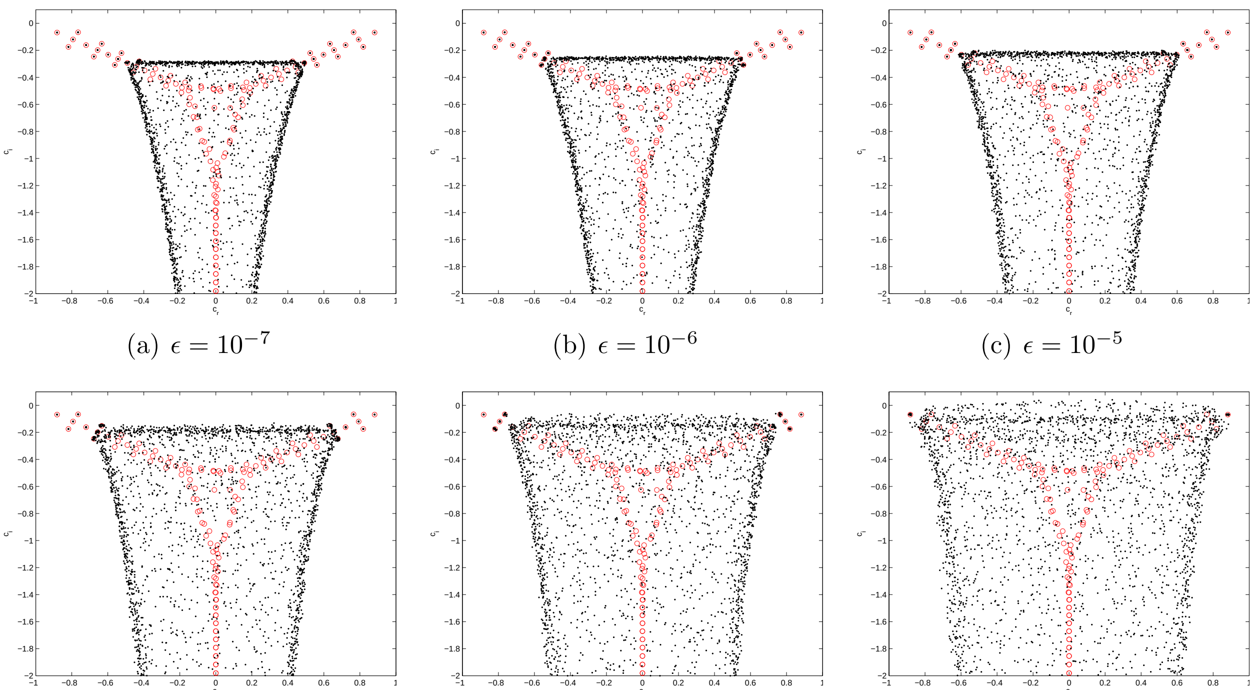
Figure 4. Eigenvalue perturbations for Poiseuille Flow with $Re = 4000$, $\alpha = 1$, $\beta = 0$

Here, there are several interesting things to note:

- Eigenvalues are most significantly sensitive near the junction of the three branches. Here, eigenvalues are moving a distance as much as 10-million times larger than the norm of the perturbation.

- Indeed, for certain values of ϵ, ϵ-pseudoeigenvalues do cross into the unstable (again, "upper") half plane. This implies both that imperfect experimental setups may experience exponential growth and that even the most ideal setups will experience significant transient growth, as per (3).

For a clearer picture, we will now utilize the first definition of the pseudospectra to create easier-to-interpret plots before drawing conclusions for transient growth expectations.



(a) $\epsilon = 10^{-7}$ (b) $\epsilon = 10^{-6}$ (c) $\epsilon = 10^{-5}$

(d) $\epsilon = 10^{-4}$ (e) $\epsilon = 10^{-3}$ (f) $\epsilon = 10^{-2}$

Figure 5. Eigenvalue perturbations for Couette Flow with $Re = 5000$, $\alpha = 1$, $\beta = 0$

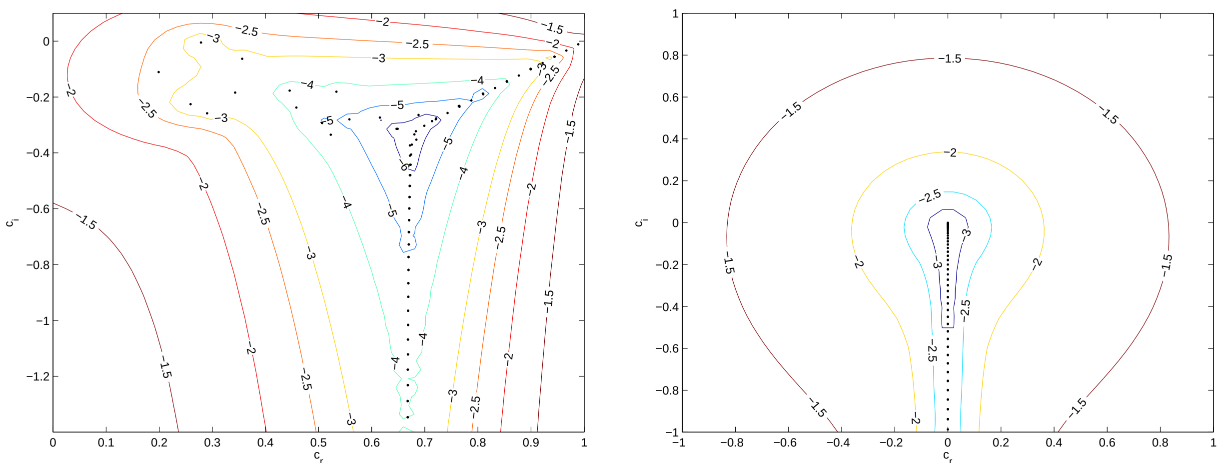
C. Pseudospectra Contours and Transient Growth Estimation

Using the first definition (1) of the pseudospectra, we can easily create a contour plot of the resolvent norm to create an overall image of the pseudospectra. To do this, we begin by defining the domain we want to consider, encompassing the unperturbed eigenvalues near the stability limit $c_i = 0$. Then, we form a grid over this portion of the complex plane, and evaluate the inverse of the resolvent norm at each point in the grid, giving ϵ directly:

$$\epsilon_{i,j} = \frac{1}{||(z_{i,j}\mathbf{I} - \mathbf{A})^{-1}||} \tag{24}$$

Then, forming the contour plot is a simple MATLAB command sequence using the built-in `contour` function.

Figures 6(a) and 6(b) display the pseudospectra contours for Poiseuille flow at a Reynolds number of 4000, for the cases $\alpha = 1$, $\beta = 0$ and $\alpha = 0$, $\beta = 1$, respectively. Labels on each contour indicate $\log_{10} \epsilon$.



(a) $\alpha = 1$, $\beta = 0$

(b) $\alpha = 0$, $\beta = 1$

Figure 6. Pseudospectra for Poiseuille Flow, Re = 4000

Using Trefethen's lower bound from Equation (3) with these plots, we can estimate the transient growth. Notice how for the $\alpha = 1$ case (2-D disturbance), the $\epsilon = 10^{-2.5}$ contour reaches into the unstable half plane a distance of approximately 0.1. Thus, we can expect transient growth as follows:

$$\max||e^{t\mathbf{A}}|| \geq \frac{\max(\text{imag}(c_\epsilon))}{\epsilon}$$

$$\max||e^{t\mathbf{A}}|| \geq \frac{0.1}{10^{-2.5}}$$

$$\max||e^{t\mathbf{A}}|| \geq 32$$

We can expect that certain initial disturbances will grow to up to 32 times their initial amplitude before decaying.

For the 3-D ($\beta = 1$) case, the situation is worse:

$$\max||e^{t\mathbf{A}}|| \geq \frac{0.17}{10^{-2.5}}$$

$$\max||e^{t\mathbf{A}}|| \geq 54$$

This is the reason it is preferrable to continue considering both 2-D and 3-D disturbances to the 2-D basic state, rather than to utilize Squires' theorem to reduce the problem to 2-D only as in (19).

We will see in the next section that after calculating the time evolution explicitly for a basic unit-norm initial condition, we will see amplifications more than twice these numbers - clearly enough to invalidate our small disturbance assumption and trigger nonlinear effects.

As an additional result, the pseudospectra for Couette flow at a Reynolds number of $Re = 5000$ are presented below in Figure 7.

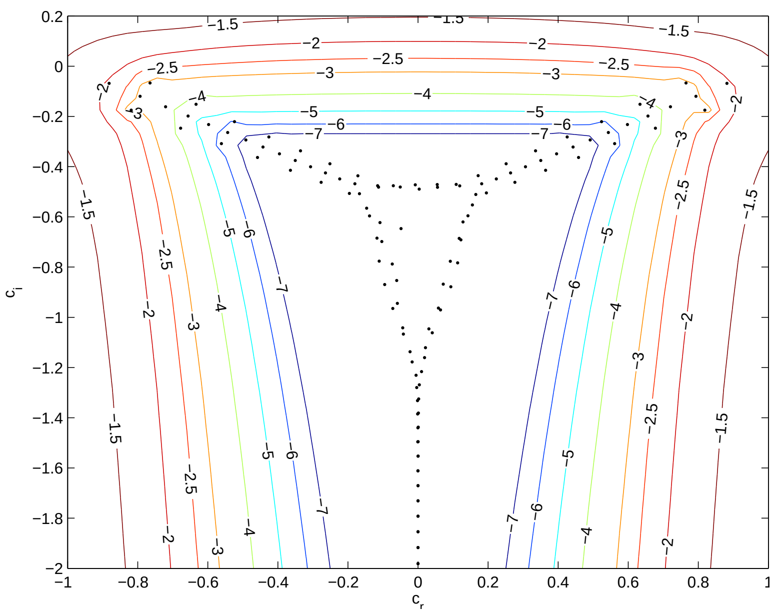


Figure 7. Pseudospectra for Couette Flow, $Re = 5000$

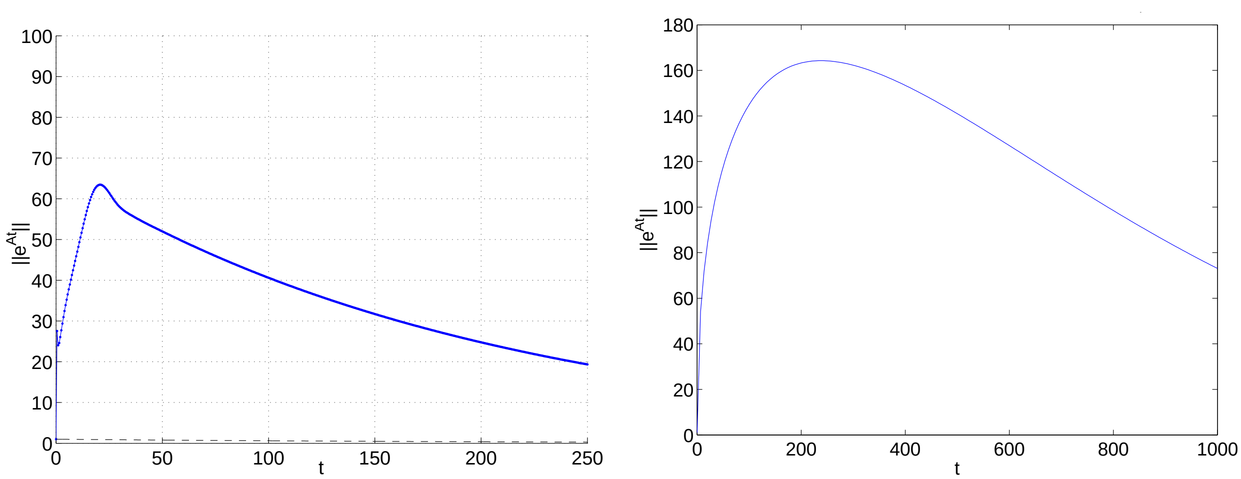
D. Transient Growth from Calculation

Now that we have defined and explored the pseudospectra method of evaluating eigenvalue sensitivity in non-normal systems, we again use our example problem of hydrodynamic stability and determine *actual* transient growth to compare with the pseudospectra-based estimation of its lower bound.

The quantity we track for transient growth is $||e^{t\mathbf{A}}||$, which represents the root-mean-square speed of the perturbation - a sort of "energy" in that it is related to the square of the velocities in the system. This curve is more like an envelope than an actual trajectory; it covers all initial conditions of unit energy norm, but at any instant, the specific initial condition that yields the value of $||e^{t\mathbf{A}}||$ may be different from the one at the next instant. Regardless, it is a very important gauge for growth of disturbances.

This quantity is straight-forward to calculate for any given time in MATLAB using `norm(expm(t*A))`, where $A = -i$ times the Orr-Sommerfeld operator, due to the form of the disturbance defined by (16).

Figure 8 below shows transient growth as calculated for the cases discussed in the previous section. Note here that, as expected, the transient growth (due to nonnormality) is large before the matrix exponential eventually decays (due to the eigenvalues). It is generally expected that such growth of several orders of magnitude begins to invalidate the linear assumption made in the derivation of the Orr-Sommerfeld operator, contributing to nonlinear effects and leading to transition to turbulence [4], [5], [6]. This can be confirmed using a greatly simplified nonlinear model proposed by Baggett and Trefethen [3].



(a) $\alpha = 1$, $\beta = 0$ (b) $\alpha = 0$, $\beta = 1$

Figure 8. Transient Growth for Poiseuille Flow, $Re = 4000$

Figure 9 is offered as an example from a parameter combination which is eigenvalue unstable. Notice that the growth here continues to grow with time, in contrast to the eigenvalue-stable cases presented in Figure 8.

Figure 10 is yet another example of transient growth – this time for Couette flow at a Reynolds number of 5000. Here, we see a growth factor of 55. Again, though Couette flow is eigenvalue stable for *all* Reynolds numbers, we see that large transient growth at this moderate Reynolds number can probably explain the extraordinary difficulty in achieving laminar Couette flow in experiment.

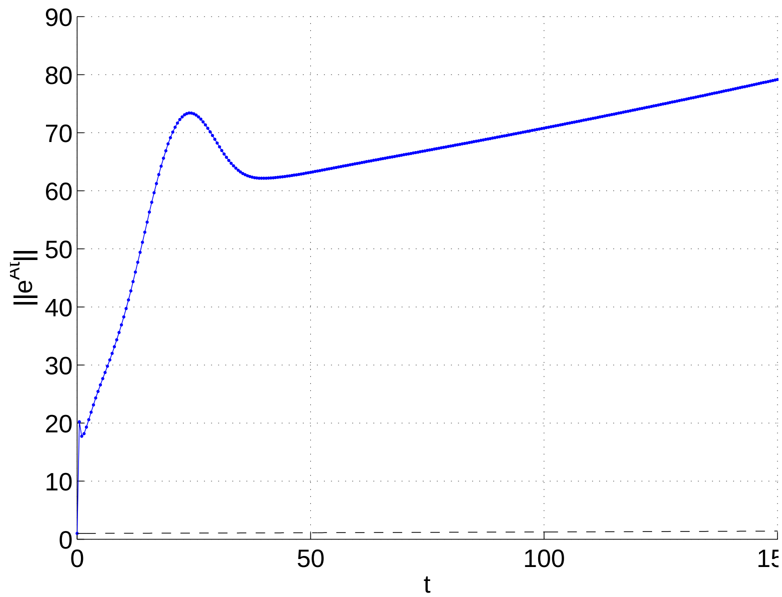


Figure 9. Transient Growth for Poiseuille Flow, $Re = 7500$

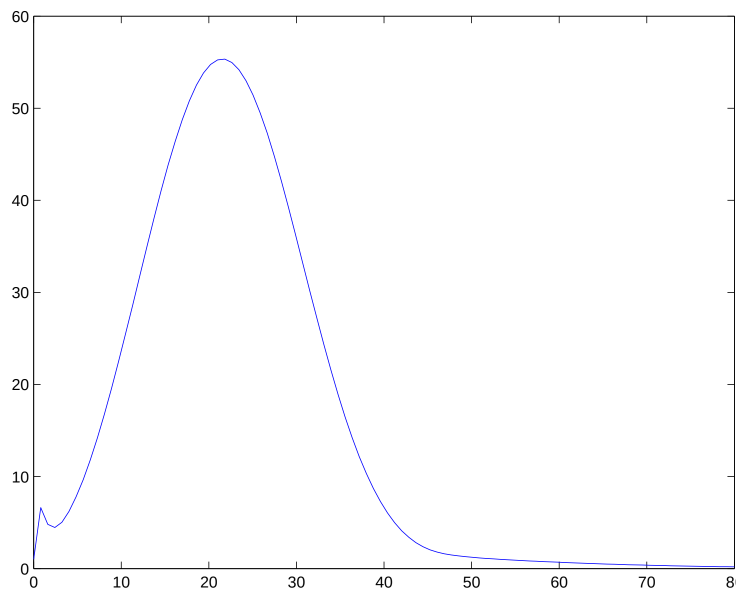


Figure 10. Transient Growth for Couette Flow, $Re = 5000$

VI. Conclusions and Recommendations

The goal of this project was to become personally familiar with transient growth as an important aspect of hydrodynamic stability, and to become experienced in the programming of a pseudospectra-based analysis tool to use on such systems. This goal has been accomplished resoundingly.

The concept of transient growth in non-normal systems is an important one for any dynamics student to understand fully. Though the more-common normal systems behave predictably in accordance with the eigenvalue analysis, it is important for an engineer to understand the situations in which such an analysis can be misleading, and to know how to analyze such systems for their potentially harmful transient behavior.

Though all results shown above were generated with custom-written codes for the purpose of learning the concepts involved, these codes are not general enough nor refined enough for public use. For this, the reader is urged to download EigTool[b] for MATLAB, written by Tom Wright at Oxford University Computing Laboratory. EigTool, shown below in Figure 11, is a fully developed, computationally efficient, easy-to-use toolbox for pseudospectral analysis and transient growth modeling. It includes several narrated examples from various fields, and is fascinating to use both as an educational tool and as applied to real problems.

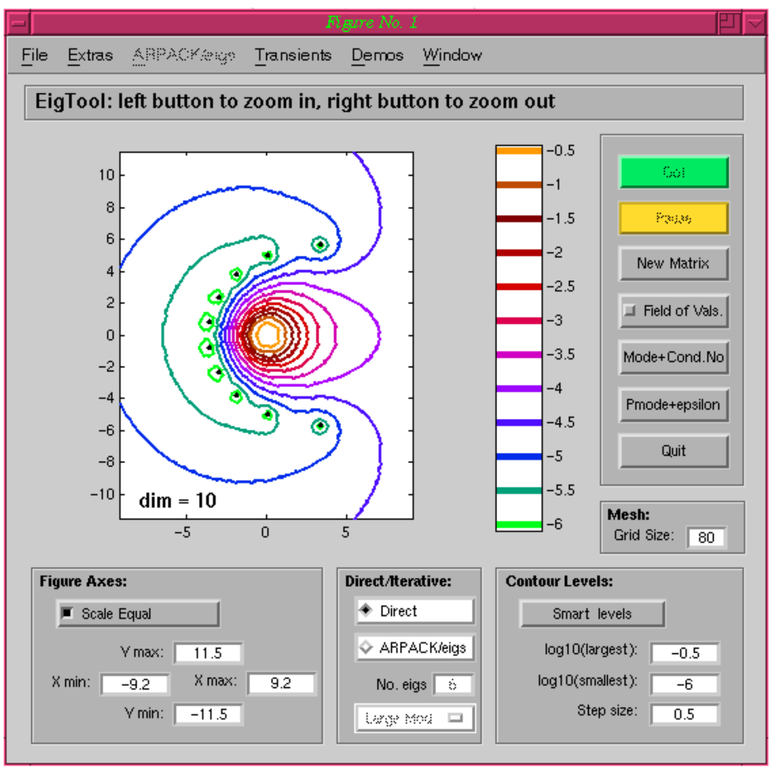


Figure 11. EigTool for MATLAB

[b] See http://web.comlab.ox.ac.uk/projects/pseudospectra/eigtool/

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